

Geometric Study of the Weak Equilibrium in a Weighted Case for a Two-Dimensional Competition Game

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ABSTRACT

In this work, an improvement of the results presented by [1] Abellanas *et al.* (Weak Equilibrium in a Spatial Model. International Journal of Game Theory, 40(3), 449-459) is discussed. Concretely, this paper investigates an abstract game of competition between two players that want to earn the maximum number of points from a finite set of points in the plane. It is assumed that the distribution of these points is not uniform, so an appropriate weight to each position is assigned. A definition of equilibrium which is weaker than the classical one is included in order to avoid the uniqueness of the equilibrium position typical of the Nash equilibrium in these kinds of games. The existence of this approximated equilibrium in the game is analyzed by means of computational geometry techniques.

Keywords: Game Theory; Approximated Equilibrium; Location; Computational Geometry; Convex Hull

1. Introduction

Works exist that generalize the study of the Nash equilibrium in a competitive game with two players [1,2]. This generalization is usually performed by means of new definitions of equilibrium that are weaker than the classical one.

In this paper, the definition of weak equilibrium and the game stated in [1] is adapted to a more general game where an appropriate weight is assigned to each position the players aim for. The results included in said paper have been then generalized to be adapted to these new considerations.

The study carried out in this article after the consideration of weights implies a step forward with respect to the existing competition models, since it enables a better adaptation to real cases.

The description of the game is as follows: Two considered players want to earn the maximum number of points from a finite set of points in the plane. For this purpose they choose their positions in the plane to attract the largest possible weight of a given set of n weighted points in the plane $H = \{p_1, \dots, p_n\}$.

The perpendicular bisector of the players' locations partitions the plane into two regions. Each player is con-

sidered to capture those points which lie closer to their own position than to that of the other player [3-8]. The pay-off of each player is then the total weight of the captured points.

In formal terms, consider a strategic game $G = (N, X, \Pi)$, where $N = \{1, 2\}$ is the set of two players, $X = \mathbb{R}^2 \times \mathbb{R}^2$ is the strategy space, and $\Pi^i : X \rightarrow \mathbb{R}$, $i = 1, 2$ is the payoff function of each player.

The description of the payoff function in the game presented above is given by:

$\Pi^1(t_1, t_2)$ weight of the points p_i such that $d(p_i, t_1) \leq d(p_i, t_2)$

$\Pi^2(t_1, t_2)$ weight of the points p_i such that $d(p_i, t_1) > d(p_i, t_2) = n - \Pi^1(t_1, t_2)$

if $t_1 \neq t_2$, where $d(p_i, t)$ represents the Euclidean distance between the points p_i, t . In the case where $t_1 = t_2$

$\Pi^1(t_1, t_2) = \Pi^2(t_1, t_2) = \frac{n}{2}$ is defined.

Therefore, if the weight of a point p_i is defined as $w(p_i) = k_i$, with $k_i > 0$ and $k_1 + \dots + k_n = n$, then the payoff of the first player will be the sum of the weights of all positions located in the same half-plane as t_1 , including the points on the bisector. The payoff of the sec-

ond player follows the same pattern, except for positions on the bisector.

Along the paper, the following definitions apply:

1) The weight of a subset $\{p_{i_1}, \dots, p_{i_k}\}$ of H is

$$w(\{p_{i_1}, \dots, p_{i_k}\}) = \sum_{j=1}^k w(p_{i_j}).$$

2) m_i stands for the i -th weight bigger than $\frac{n}{2}$ that can be attained by a subset of H . Therefore m_i is recursively defined as follows:

$$m_1 = \min_{I \subset H} \left\{ w(I) / w(I) > \frac{n}{2} \right\} \text{ and}$$

$$m_i = \min_{I \subset H} \left\{ w(I) / w(I) > m_{i-1} \right\} \text{ for } i \geq 2.$$

3) C_α is the intersection of the convex hulls of the subsets of H with weight bigger than α . For a definition of the convex hull of a set, see [9].

The rest of this paper is structured as follows: In Section 2, necessary and sufficient conditions are presented for the weak equilibrium positions to exist in this weighted game. Section 3 studies the maximum number of weak equilibrium positions in which the two players are located in positions of H .

2. Weak Equilibrium

Lillo *et al.* 2007 [10] present necessary and sufficient conditions for the Nash equilibrium positions to exist in a game as the one considered in this paper. These conditions are stated in terms of geometric features such as convex hulls. The main conclusion is that the Nash equilibrium in the game, if it exists, is a unique point (t, t) for some $t \in \mathbb{R}^2$. In other words, both players will choose the same position, except in cases where the points of H are aligned.

To avoid this situation, a definition of equilibrium which is weaker than the classical one is proposed:

Definition 1: A strategy profile (t_1^0, t_2^0) is a weak equilibrium if:

$$\Pi^1(t_1, t_2^0) \leq K_1, \Pi^2(t_1^0, t_2) \leq K_2 \quad \forall t_1, t_2 \in \mathbb{R}^2 \text{ where}$$

$$K_1 = \min_{I \subset H} \left\{ w(I) / w(I) > \Pi^1(t_1^0, t_2^0) \right\} \text{ and}$$

$$K_2 = \min_{I \subset H} \left\{ w(I) / w(I) > \Pi^2(t_1^0, t_2^0) \right\}, \text{ that is to say}$$

K_1 is the least weight greater than $\Pi^1(t_1^0, t_2^0)$ of a subset of H (analogously for K_2).

Remark: This definition is an adaptation to the discrete case of the ε -equilibrium studied by Monderer and Shapley, 1996 [2].

A geometric analysis is developed here that extends the concepts relative to the study of the Nash equilibrium in a weighted game and underlies the search for the equilibrium positions, if they exist, according to the new definition.

Proposition 1:

A weak equilibrium position (t_1^0, t_2^0) necessarily satisfies $\Pi^1(t_1^0, t_2^0) \geq n - m_1, \Pi^2(t_1^0, t_2^0) \geq n - m_1$.

Proof:

If there exists a weak equilibrium position with, say, $\Pi^1(t_1^0, t_2^0) < n - m_1$, then it is satisfied that

$$\Pi^1(t_2^0, t_2^0) = \frac{n}{2} > n - m_1 > \Pi^1(t_1^0, t_2^0), \text{ so } \Pi^1(t_2^0, t_2^0) > K_1 \text{ a}$$

contradiction because (t_1^0, t_2^0) is a weak equilibrium position.

Remark: The proposition yields that the payoffs in a weak equilibrium position must be $n - m_1$ for a player and m_1 for the other or $\frac{n}{2}$ for both. It can be noted

that $n - m_1$ is the largest payoff smaller than $\frac{n}{2}$ that a player can obtain.

The following proposition gives a necessary and sufficient condition for the existence of weak equilibrium positions:

Proposition 2:

If $m_1 < n$, then there exists weak equilibrium positions in the proposed game if and only if $C_{m_1} \neq \emptyset$.

Proof:

First it is seen that the condition is necessary: If $C_{m_1} = \emptyset$, then there exists a subset of H with weight greater than m_1 such that its convex hull does not contain the position t_2^0 chosen by the second player (assumed without loss of generality that the payoff of the second player in said position is greater or equal than $\frac{n}{2}$). So if the first player is located in the position t_1^0

with a payoff lower than or equal to $\frac{n}{2}$, then the first

player can deviate to a position t_1 that separates t_2^0 and the convex hull aforementioned, hence the first player will get a payoff greater than m_1 , therefore $\Pi^1(t_1, t_2^0) > K_1$. Consequently, there is not a position (t_1^0, t_2^0) of weak equilibrium.

Now it is stated that the condition is sufficient: If C_{m_1} is not the empty set, then any player can locate in C_{m_1} to ensure that the other player cannot get an earning bigger than m_1 in any location, because the first player controls all the subsets of H with weight bigger than m_1 . Hence, every position (t, t) with t in C_{m_1} is a weak equilibrium position, because $\Pi^i(t, t) = \frac{n}{2}$ for $i = 1, 2$

and any player who deviates would obtain a payoff smaller or equal to m_1 .

Now propositions 1 and 2 are applied to describe the

weak equilibrium positions. The possible weak equilibrium positions in the proposed game are:

1) (t_1, t_2) with t_1, t_2 belonging to C_{m_1} ,

$$\Pi^i(t_1, t_2) = \frac{n}{2} \text{ for } i=1, 2.$$

2) If there does not exist a combination of points of H with weight $\frac{n}{2}$, the positions (t_1, t_2) with one of the points, say t_1 , belonging to C_{m_1} , the other one, t_2 , belonging to C_{m_2} , $\Pi^1(t_1, t_2) = m_1$ and $\Pi^2(t_1, t_2) = n - m_1$.

3) If there exists a combination of points of H with weight $\frac{n}{2}$, the positions (t_1, t_2) with one of the points, say t_1 , belonging to C_{m_2} , the other point, t_2 , belonging to C_n , $\Pi^1(t_1, t_2) = n - m_1$ and $\Pi^2(t_1, t_2) = m_1$.

3. Maximum Number of Weak Equilibrium Positions

Now the problem of finding the maximum number of points of H that can be included in C_{m_1} will be considered. This subset of H can be interesting for practical purposes, because it carries a large number of weak equilibrium positions in which at least one player locates in the position of one of the points in H . As an example, if the game has a political scope, where the players are parties and H is a finite set of types of voters, then this subset gives a large number of weak equilibrium positions with the parties fitting to one type of voters.

Concretely the following problem is considered:

Find the maximum number of points of H that can be included in C_{m_1} for all the possible configurations of n points in the plane (no three aligned) and all the possible assignments of weights for these n points with $m_2 < n$. This last condition implies that $n > 2$. The maximum number to be found is denoted by \max .

3.1. An Upper Bound

The following proposition gives an upper bound for \max that is bounded in n . A preliminary lemma is needed:

Lemma 1:

There is some point of H in the boundary of the convex hull of H that does not belong to C_{m_1} .

Proof:

If $n = 3$, then the points of H are in convex position (a point set is in convex position if every point of the set is a vertex of its convex hull), so the condition $m_2 < n$ implies that there is a convex hull of a subset of H of weight $> m_1$ that does not contain every point of H . Hence there is a point of H in the boundary of the convex hull of H that does not belong to C_{m_1} .

If $n = 4$ and every point of H in the boundary of the convex hull of H belongs to C_{m_1} , then every subset of H

with two elements must have weight $\leq \frac{n}{2} = 2$. Otherwise,

there would be a subset of H of cardinal three with weight $> m_1$, not containing every point of H in the boundary of the convex hull of H , a contradiction because C_{m_1} contains every point of H in the boundary of the convex hull. With 4 being the weight of H , this implies that every subset of H with two elements must have weight $= 2$. But this is possible if and only if $w(p_i) = 1$ for every $p_i \in H$ and then $m_2 = 4 = n$, a contradiction.

If $n > 4$ and every point of H in the boundary of the convex hull of H belongs to C_{m_1} , then $w(p_i) \geq n - m_1$ for every $p_i \in H$ such that p_i is in the boundary of the convex hull of H . Hence, if the four points of H with smallest x coordinate are considered, say p_1, p_2, p_3, p_4 , then C_{m_1} is included in the convex hull of $\{p_1, p_2, p_3, p_4\}$. So there is at least a point of H in the boundary of the convex hull of H that does not belong to C_{m_1} , which is the point of H with the greatest x coordinate, a contradiction.

Proposition 3:

It is satisfied that $\max \leq 3$.

Proof:

Consider a point of H in the boundary of the convex hull of H and not belonging to C_{m_1} , say p_n , which exists by lemma 1. Arrange the other points of H according to the angle with respect to p_n , say p_1, \dots, p_{n-1} . The initial angle 0 is determined by a ray emanating from p_n and containing a segment of the boundary of the convex hull of H . Hence, if i is the smallest index such that

$\{p_1, \dots, p_i\}$ has weight greater than $\frac{n}{2}$, then

$$w(\{p_1, \dots, p_i, p_n\}) > m_1, \quad w(\{p_1, \dots, p_{i-1}\}) \leq \frac{n}{2}, \text{ so}$$

$w(\{p_i, \dots, p_{n-1}, p_n\}) \geq \frac{n}{2}$. This last inequality implies that

$w(\{p_{i-2}, p_{i-1}, p_i, \dots, p_{n-1}, p_n\}) > m_1$, so C_{m_1} is included in the intersection of the convex hulls of $\{p_1, \dots, p_i, p_n\}$, $\{p_{i-2}, p_{i-1}, p_i, \dots, p_{n-1}, p_n\}$. This intersection is included in the region limited by the ray emanating from p_n to p_{i-2} and the ray emanating from p_n to p_i . This region contains four points of H : p_{i-2}, p_{i-1}, p_i and p_n .

But p_n is not in C_{m_1} by the assumption, so at most three points of H can be located in C_{m_1} and then $\max \leq 3$. Note that if $i \leq 2$, then p_{i-2} makes no sense. But in this case C_{m_1} is included in the convex hull of $\{p_1, p_2, p_n\}$, so at most two points of H are included in C_{m_1} .

Remark: If $n = 3$ and $m_2 < n = 3$, then a point of H , say p_1 , must have weight m_1 . Hence C_{m_1} is contained in the intersection of the convex hulls of $\{p_1, p_2\}$, $\{p_1, p_3\}$, that is to say, in $\{p_1\}$. As a matter of fact, $C_{m_1} = \{p_1\}$ because every subset of H with weight

greater than m_1 contains to p_1 . So $\max = 1$ for $n = 3$.

3.2. Examples Where the Upper Bound Is Attained

Now it can be seen that for even n , $n > 4$, the upper bound for \max is attained in an example with equally weighted points, so $\max = 3$ for even n , $n > 4$:

Example 1:

Let n be an even number greater than 4 and consider $n-3$ points p_1, \dots, p_{n-3} in convex position. If three new points p_{n-2}, p_{n-1}, p_n are located in the intersection $C_{(n-3),2}$ of the convex hulls of the subsets of

$\{p_1, \dots, p_{n-3}\}$ with $\left\lceil \frac{n-3}{2} \right\rceil + 2$ points, in such way that

the n points are in general position (no three points in the same line) and weight 1 is assigned to each point, then a set $H = \{p_1, \dots, p_n\}$ with three points in C_{m_1} (p_{n-2}, p_{n-1}, p_n) is obtained.

An example with 3 points in C_{m_1} can be seen for odd n , $n > 5$, so $\max = 3$ for odd n , $n > 5$:

Example 2:

Consider $n-3$ points p_1, \dots, p_{n-3} with weight

$\frac{n}{n+1}$ each in the vertices of a regular polygon, a point

p_{n-2} with weight $\frac{2n}{n+1}$ in $C_{(n-3),2}$ and two points

p_{n-1}, p_n with weight $\frac{n}{n+1}$ each and in $C_{(n-2),2}$. First

it is noted that p_{n-2} can be located in the intersection aforementioned and non-collinear with two points of $\{p_1, \dots, p_{n-3}\}$ because that intersection does not have an empty interior set. In the same way it can be seen that the points p_{n-1}, p_n can be located in $C_{(n-2),2}$ such that the n points are in general position because this last intersection also does not have an empty interior set.

Now it can be seen that there does exist a subset of $H = \{p_1, \dots, p_n\}$ with weight $\frac{n}{2}$. If it is considered

$k = \frac{n+1}{2}$ points of weight $\frac{n}{n+1}$, then the weight of the

subset is $k \frac{n}{n+1} = \frac{n}{2}$. Hence a subset of H with weight

$> m_1$ must contain $\frac{n+1}{2} + 2$ points of weight $\frac{n}{n+1}$ or

$\frac{n+1}{2}$ points of weight $\frac{n}{n+1}$ and p_{n-2} . In this last

case, if said set of points does not contain both $p_{n-1},$

p_n , then it must contain at least $\frac{n-3}{2} + 1$ points from

$\{p_1, \dots, p_{n-3}\}$ and p_{n-2} , so its convex hull contains

p_{n-1}, p_n by the construction of the set H . In the former

case, the said set of points contains at least $\frac{n-3}{2} + 2$

points from $\{p_1, \dots, p_{n-3}\}$, so its convex hull contains p_{n-2}, p_{n-1}, p_n by the construction of the set H . Consequently, p_{n-2}, p_{n-1}, p_n belong to C_{m_1} as desired.

Now examples of sets H with three points in C_{m_1} are shown for $n = 4, 5$:

Example 3:

Consider three non-collinear points p_1, p_2, p_3 and a point p_4 inside the triangle having vertices p_1, p_2, p_3 . Assign weight $\frac{4}{3}$ to p_1, p_3 , $\frac{2}{3}$ to p_2, p_4 . See

Figure 1. This allows for $m_1 = \frac{8}{3}$, which is attained in

the following subsets of $H = \{p_1, \dots, p_4\}$: $\{p_1, p_2, p_4\}$, $\{p_2, p_3, p_4\}$, $\{p_1, p_3\}$. Hence the subsets of H with weight $> \frac{8}{3}$ must contain p_1, p_3 and one point of

$\{p_2, p_4\}$, and then their convex hulls contain the convex hull of $\{p_1, p_3, p_4\}$. This implies that C_{m_1} contains $\{p_1, p_3, p_4\}$.

Example 4:

Consider three non-collinear points p_1, p_2, p_3 and two points p_4, p_5 inside the triangle having vertices

p_1, p_2, p_3 . Assign weight $\frac{5}{3}$ to p_1 , $\frac{5}{6}$ to $p_2, \dots,$

p_5 . See **Figure 2.** This allows for $m_1 = \frac{10}{3}$, which is

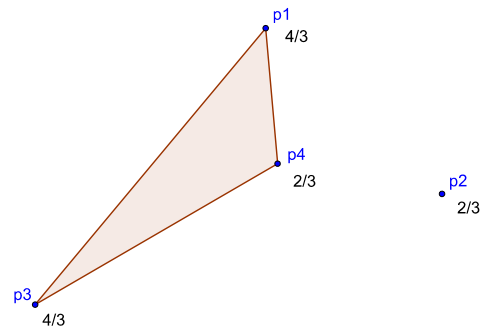


Figure 1. Example for $n = 4$. C_{m_1} is the shaded region.

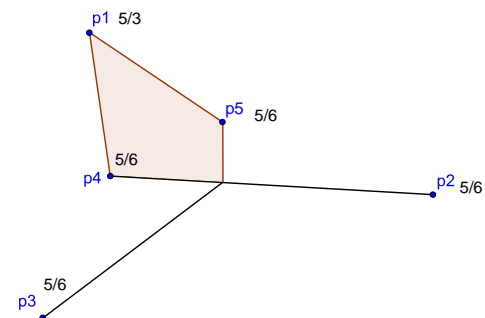


Figure 2. Example for $n = 5$. C_{m_1} is the shaded region.

attained in the subsets of $H = \{p_1, \dots, p_5\}$ of cardinal three containing p_1 and in the set $\{p_2, p_3, p_4, p_5\}$. Hence the subsets of H with weight $> \frac{10}{3}$ must con-

tain to p_1 and at least three points of $\{p_2, p_3, p_4, p_5\}$, and then their convex hulls contain $\{p_1, p_4, p_5\}$. This implies that C_{m_1} contains this set of three points of H .

Proposition 3, its remark and the previous examples yield the following corollary:

Corollary 1:

It is satisfied that $\max = 1$ for $n > 3$.

4. Conclusions

A discrete two-dimensional competition model has been proposed and analysed using geometric strategies. To give the game a general scope, a weight to each point of the finite set which the players fight for has been assigned. In this model, Nash equilibrium positions in the majority of the situations do not exist, see [10]. To resolve this situation, a weakened definition of equilibrium has been presented, which ensures for each player that the other can not improve his payoff by more than one quantity if he changes his position. This new definition of equilibrium can be useful in cases which have no Nash equilibrium.

These considerations mean a step forward in the competition models existing in the literature and adapt better to real situations. For instance, if the game is applied to the optimum location of services to gain customers, assigning weights to the different customers implies the consideration that those who are potentially more related to the service can have a higher value in the competition.

The study of the existence and locations of weak equilibrium positions in this paper have been expanded in scope by applying techniques from computational geometry such as the intersection of convex hulls, which can be used because of the discrete nature of the game.

With respect to the study of the maximum number of weak equilibrium positions located in the set H of target points, it has been able to find that maximum for all the sets of n points and all the possible weightings of such points, thus obtaining a very general and original result. Furthermore, the presented examples show that the cases

in which said maximum number is attained are the ones with equally or almost equally weighted points. Therefore, these are the most interesting cases for applied purposes.

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